

ON BOUNDARY BEHAVIOR OF SPATIAL MAPPINGS

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Abstract

We show that homeomorphisms f in \mathbb{R}^n , $n \geq 3$, of finite distortion in the Orlicz–Sobolev classes $W_{\text{loc}}^{1,\varphi}$ with a condition on φ of the Calderon type and, in particular, in the Sobolev classes $W_{\text{loc}}^{1,p}$ for $p > n - 1$ are the so-called lower Q -homeomorphisms, $Q(x) = K_I^{\frac{1}{n-1}}(x, f)$, where $K_I(x, f)$ is its inner dilatation. The statement is valid also for all finitely bi-Lipschitz mappings that a far-reaching extension of the well-known classes of isometric and quasiisometric mappings. This makes possible to apply our theory of the boundary behavior of the lower Q -homeomorphisms to all given classes.

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1 Introduction

The problem of the boundary behavior is one of the central topics of the theory of quasiconformal mappings and their generalizations. At present mappings with finite distortion are studied, see many references in the monographs [12] and [30]. In this case, as it was earlier, the main geometric approach in the modern mapping theory is the method of moduli, see, e.g., the monographs [12], [30], [35], [43], [52], [53] and [55].

It is well-known that the concept of moduli with weights essentially due to Andreian Cazacu, see, e.g., [1]–[3], see also recent works [6]–[8] of her learner. At the present paper we give new modulus estimates for space mappings that essentially improve the corresponding estimates first obtained in the paper [23].

Here we apply our theory of the so-called lower Q -homeomorphisms first introduced in the paper [20], see also the monograph [30], that was motivated by the ring definition of quasiconformal mappings of Gehring, see [10]. The theory of lower Q -homeomorphisms has already found interesting applications to the theory of the Beltrami equations in the plane and to the theory of mappings of the classes of Sobolev and Orlich-Sobolev in the space, see, e.g., [17], [18], [23], [24], [25], [30] and [45].

Following Orlicz, see, e.g., paper [36], see also monograph [58], given a convex increasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\varphi(0) = 0$, denote by L^φ the space of all functions $f : D \rightarrow \mathbb{R}$ such that

$$\int_D \varphi \left(\frac{|f(x)|}{\lambda} \right) dm(x) < \infty \quad (1.1)$$

for some $\lambda > 0$ where $dm(x)$ corresponds to the Lebesgue measure in D . L^φ is called the **Orlicz space**. In other words, L^φ is the cone over the class of all functions $g : D \rightarrow \mathbb{R}$ such that

$$\int_D \varphi (|g(x)|) dm(x) < \infty \quad (1.2)$$

which is also called the **Orlicz class**, see [4].

The **Orlicz–Sobolev class** $W^{1,\varphi}(D)$ is the class of all functions $f \in L^1(D)$ with the first distributional derivatives whose gradient ∇f belongs to the Orlicz class in D . $f \in W_{\text{loc}}^{1,\varphi}(D)$ if $f \in W^{1,\varphi}(D_*)$ for every domain D_* with a compact closure in D . Note that by definition $W_{\text{loc}}^{1,\varphi} \subseteq W_{\text{loc}}^{1,1}$. As usual, we write $f \in W_{\text{loc}}^{1,p}$ if $\varphi(t) = t^p$, $p \geq 1$. Later on, we also write $f \in W_{\text{loc}}^{1,\varphi}$ for a locally integrable vector-function $f = (f_1, \dots, f_m)$ of n real variables x_1, \dots, x_n if $f_i \in W_{\text{loc}}^{1,1}$ and

$$\int_D \varphi (|\nabla f(x)|) dm(x) < \infty \quad (1.3)$$

where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left(\frac{\partial f_i}{\partial x_j} \right)^2}$. Note that in this paper we use the notation $W_{\text{loc}}^{1,\varphi}$ for more general functions φ than in those classic Orlicz classes often giving up

the conditions on convexity and normalization of φ . Note also that the Orlicz–Sobolev classes are intensively studied in various aspects at the moment, see, e.g., [24] and references therein.

In this connection, recall definitions which are relative to Sobolev's classes. Given an open set U in \mathbb{R}^n , $n \geq 2$, $C_0^\infty(U)$ denotes the collection of all functions $\psi : U \rightarrow \mathbb{R}$ with compact support having continuous partial derivatives of any order. Now, let u and $v : U \rightarrow \mathbb{R}$ be locally integrable functions. The function v is called the **distributional (generalized) derivative** u_{x_i} of u in the variable x_i , $i = 1, 2, \dots, n$, $x = (x_1, x_2, \dots, x_n)$, if

$$\int_U u \psi_{x_i} dm(x) = - \int_U v \psi dm(x) \quad \forall \psi \in C_0^\infty(U). \quad (1.4)$$

$u \in W_{\text{loc}}^{1,1}$ if u_{x_i} exist for all $i = 1, 2, \dots, n$. These concepts were introduced by Sobolev in \mathbb{R}^n , $n \geq 2$, see [51], and at present it is developed under wider settings by many authors, see, e.g., [42] and further references in [24].

Recall also the following topological notion. A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is said to be **locally connected at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 , there is a neighborhood $V \subseteq U$ of x_0 such that $V \cap D$ is connected. Note that every Jordan domain D in \mathbb{R}^n is locally connected at each point of ∂D , see, e.g., [57], p. 66.

Following [19] and [20], see also [30] and [44], we say that ∂D is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there is a neighborhood $V \subset U$ of x_0 such that

$$M(\Delta(E, F, D)) \geq P \quad (1.5)$$

for all continua E and F in D intersecting ∂U and ∂V . Here M is the modulus, see (3.5), and $\Delta(E, F, D)$ the family of all paths $\gamma : [a, b] \rightarrow \overline{\mathbb{R}^n}$ connecting E and F in D , i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$. We say that the boundary ∂D is **weakly flat** if it is weakly flat at every point in ∂D .

We also say that a **point** $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood U of the point x_0 , there exist a compactum E in D , a neighborhood $V \subset U$ of x_0 and a number $\delta > 0$ such that

$$M(\Delta(E, F, D)) \geq \delta \quad (1.6)$$

for all continua F in D intersecting ∂U and ∂V . We say that the **boundary** ∂D is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.

It is easy to see that if a domain D in \mathbb{R}^n is weakly flat at a point $x_0 \in \partial D$, then the point x_0 is strongly accessible from D . Moreover, it was proved by us that if a domain D in \mathbb{R}^n is weakly flat at a point $x_0 \in \partial D$, then D is locally connected at x_0 , see, e.g., Lemma 5.1 in [20] or Lemma 3.15 in [30].

The notions of strong accessibility and weak flatness at boundary points of a domain in \mathbb{R}^n defined in [19], see also [20], are localizations and generalizations of the corresponding notions introduced in [28]–[29], cf. with properties P_1 and P_2 by Väisälä in [53] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [34]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary.

In the mapping theory and in the theory of differential equations, it is often applied the so-called Lipschitz domains whose boundaries are locally quasiconformal. Recall first that a map $\varphi : X \rightarrow Y$ between metric spaces X and Y is said to be **Lipschitz** provided $\text{dist}(\varphi(x_1), \varphi(x_2)) \leq M \cdot \text{dist}(x_1, x_2)$ for some $M < \infty$ and for all x_1 and $x_2 \in X$. The map φ is called **bi-Lipschitz** if, in addition, $M^* \text{dist}(x_1, x_2) \leq \text{dist}(\varphi(x_1), \varphi(x_2))$ for some $M^* > 0$ and for all x_1 and $x_2 \in X$. Later on, X and Y are subsets of \mathbb{R}^n with the Euclidean distance.

It is said that a bounded domain D in \mathbb{R}^n is **Lipschitz** if every point $x_0 \in \partial D$ has a neighborhood U that can be mapped by a bi-Lipschitz homeomorphism φ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ in such a way that $\varphi(\partial D \cap U)$ is the intersection of \mathbb{B}^n with the a coordinate hyperplane and $f(x_0) = 0$, see, e.g., [35]. Note that a bi-Lipschitz homeomorphism is quasiconformal and the modulus is a quasiinvariant under such mappings. Hence the Lipschitz domains have weakly flat boundaries. In particular, smooth and convex bounded domains are so.

2 On BMO, VMO and FMO functions

The BMO space was introduced by John and Nirenberg in [16] and soon became one of the main concepts in harmonic analysis, complex analysis, partial differential equations and related areas, see, e.g., [13] and [40].

Let D be a domain in \mathbb{R}^n , $n \geq 1$. Recall that a real valued function $\varphi \in L^1_{\text{loc}}(D)$ is said to be of **bounded mean oscillation** in D , abbr. $\varphi \in \text{BMO}(D)$ or simply $\varphi \in \text{BMO}$, if

$$\|\varphi\|_* = \sup_{B \subset D} \int_B |\varphi(z) - \varphi_B| dm(z) < \infty \quad (2.1)$$

where the supremum is taken over all balls B in D and

$$\varphi_B = \int_B \varphi(z) dm(z) = \frac{1}{|B|} \int_B \varphi(z) dm(z) \quad (2.2)$$

is the mean value of the function φ over B . Note that $L^\infty(D) \subset \text{BMO}(D) \subset L^p_{\text{loc}}(D)$ for all $1 \leq p < \infty$, see, e.g., [40].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in \text{VMO}$, if the supremum in (2.1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \rightarrow 0$. VMO has been introduced by Sarason in [50]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see, e.g., [5], [15], [31], [37] and [39].

Following [14], we say that a function $\varphi : D \rightarrow \mathbb{R}$ has **finite mean oscillation at a point** $z_0 \in D$, write $\varphi \in \text{FMO}(x_0)$, if

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \tilde{\varphi}_\varepsilon(z_0)| dm(z) < \infty \quad (2.3)$$

where

$$\tilde{\varphi}_\varepsilon(z_0) = \int_{B(z_0, \varepsilon)} \varphi(z) dm(z) \quad (2.4)$$

is the mean value of the function $\varphi(z)$ over the ball $B(z_0, \varepsilon)$. Condition (2.3) includes the assumption that φ is integrable in some neighborhood of the point z_0 . By the triangle inequality, we obtain the following statement.

Proposition 2.1. *If for some collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi_\varepsilon| dm(z) < \infty, \quad (2.5)$$

then φ is of finite mean oscillation at z_0 .

Choosing in Proposition 2.1 $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$, we have the following.

Corollary 2.1. *If for a point $z_0 \in D$*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z)| dm(z) < \infty, \quad (2.6)$$

then φ has finite mean oscillation at z_0 .

Recall that a point $z_0 \in D$ is called a Lebesgue point of a function $\varphi : D \rightarrow \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \rightarrow 0} \int_{B(z_0, \varepsilon)} |\varphi(z) - \varphi(z_0)| dm(z) = 0. \quad (2.7)$$

It is known that almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus, we have the following conclusion.

Corollary 2.2. *Every function $\varphi : D \rightarrow \mathbb{R}$ which is locally integrable, has a finite mean oscillation at almost every point in D .*

Remark 2.1. Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk Δ , see, e.g., [40], p. 5, and hence also to FMO. However, $\tilde{\varphi}_\varepsilon(0) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, showing that the condition (2.6) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0 .

Clearly that $\text{BMO} \subset \text{FMO} \subset L^1_{\text{loc}}$ but FMO is not a subset of L^p_{loc} for any $p > 1$, see examples in Section 11.2 of the monograph [30], in comparison with $\text{BMO}_{\text{loc}} \subset L^p_{\text{loc}}$ for all $p \in [1, \infty)$. Thus, FMO is essentially wider than BMO_{loc} . The following lemma is key, see Corollary 2.3 in [14] and Corollary 6.3 in [30].

Lemma 2.1. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, $x_0 \in D$, and let $\varphi : D \rightarrow \mathbb{R}$ be a non-negative function of the class $\text{FMO}(x_0)$. Then*

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} \frac{\varphi(x) dm(x)}{\left(|x - x_0| \log \frac{1}{|x - x_0|}\right)^n} = O\left(\log \log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad (2.8)$$

for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup_{x \in D} |x - x_0|$.

3 On lower Q -homeomorphisms

Let ω be an open set in $\overline{\mathbb{R}^k}$, $k = 1, \dots, n-1$. Recall that a (continuous) mapping $S : \omega \rightarrow \mathbb{R}^n$ is called a k -dimensional surface S in \mathbb{R}^n . The number of preimages

$$N(S, y) = \text{card } S^{-1}(y) = \text{card } \{x \in \omega : S(x) = y\}, \quad y \in \mathbb{R}^n \quad (3.1)$$

is said to be a **multiplicity function** of the surface S . It is known that the multiplicity function is lower semicontinuous, i.e.,

$$N(S, y) \geq \liminf_{m \rightarrow \infty} N(S, y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \dots$, such that $y_m \rightarrow y \in \mathbb{R}^n$ as $m \rightarrow \infty$, see, e.g., [38], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure H^k , see, e.g., [49], p. 52.

Recall that a k -dimensional Hausdorff area in \mathbb{R}^n (or simply **area**) associated with a surface $S : \omega \rightarrow \mathbb{R}^n$ is given by

$$\mathcal{A}_S(B) = \mathcal{A}_S^k(B) := \int_B N(S, y) dH^k y \quad (3.2)$$

for every Borel set $B \subseteq \mathbb{R}^n$ and, more generally, for an arbitrary set that is measurable with respect to H^k in \mathbb{R}^n , cf. 3.2.1 in [9] and 9.2 in [30].

If $\varrho : \mathbb{R}^n \rightarrow \overline{\mathbb{R}^+}$ is a Borel function, then its **integral over** S is defined by the equality

$$\int_S \varrho d\mathcal{A} := \int_{\mathbb{R}^n} \varrho(y) N(S, y) dH^k y. \quad (3.3)$$

Given a family Γ of k -dimensional surfaces S , a Borel function $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$ is called **admissible** for Γ , abbr. $\varrho \in \text{adm } \Gamma$, if

$$\int_S \varrho^k d\mathcal{A} \geq 1 \quad (3.4)$$

for every $S \in \Gamma$. The **modulus** of Γ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \varrho^n(x) dm(x). \quad (3.5)$$

We also say that a Lebesgue measurable function $\varrho : \mathbb{R}^n \rightarrow [0, \infty]$ is **extensively admissible** for a family Γ of k -dimensional surfaces S in \mathbb{R}^n , abbr. $\varrho \in \text{ext adm } \Gamma$, if a subfamily of all surfaces S in Γ , for which (3.4) fails, has the modulus zero.

Given domains D and D' in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geq 2$, $x_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : \mathbb{R}^n \rightarrow (0, \infty)$, we say that a homeomorphism $f : D \rightarrow D'$ is a **lower Q -homeomorphism at the point x_0** if

$$M(f\Sigma_\varepsilon) \geq \inf_{\varrho \in \text{ext adm } \Sigma_\varepsilon} \int_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)} dm(x) \quad (3.6)$$

for every ring $R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}$, $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 \in (0, d_0)$, where $d_0 = \sup_{x \in D} |x - x_0|$, and Σ_ε denotes the family of all intersections of the spheres $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $r \in (\varepsilon, \varepsilon_0)$, with D .

The notion of lower Q -homeomorphisms in the standard way can be extended to ∞ through inversions. We also say that a homeomorphism $f : D \rightarrow \overline{\mathbb{R}^n}$ is a **lower Q -homeomorphism on ∂D** if f is a lower Q -homeomorphism at every point $x_0 \in \partial D$.

We proved the following significant statements on lower Q -homeomorphisms, see Theorem 10.1 (Lemma 6.1) in [20] or Theorem 9.8 (Lemma 9.4) in [30].

Proposition 3.1. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 2$, $Q : D \rightarrow (0, \infty)$ a measurable function and $f : D \rightarrow D'$ a lower Q -homeomorphism on ∂D . Suppose that the domain D is locally connected on ∂D and that the domain D' has a (strongly accessible) weakly flat boundary. If*

$$\int_0^{\delta(x_0)} \frac{dr}{\|Q\|_{n-1}(x_0, r)} = \infty \quad \forall x_0 \in \partial D \quad (3.7)$$

for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$\|Q\|_{n-1}(x_0, r) = \left(\int_{D \cap S(x_0, r)} Q^{n-1}(x) d\mathcal{A} \right)^{\frac{1}{n-1}},$$

then f can be extended to a (continuous) homeomorphic mapping $\overline{f} : \overline{D} \rightarrow \overline{D}'$.

4 A connection with the Orlicz–Sobolev classes

Given a mapping $f : D \rightarrow \mathbb{R}^n$ with partial derivatives a.e., recall that $f'(x)$ denotes the Jacobian matrix of f at $x \in D$ if it exists, $J(x) = J(x, f) = \det f'(x)$ is the Jacobian of f at x , and $\|f'(x)\|$ is the operator norm of $f'(x)$, i.e.,

$$\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}. \quad (4.1)$$

We also let

$$l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}. \quad (4.2)$$

The **outer dilatation** of f at x is defined by

$$K_O(x) = K_O(x, f) = \begin{cases} \frac{\|f'(x)\|^n}{|J(x, f)|} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } f'(x) = 0, \\ \infty & \text{otherwise,} \end{cases} \quad (4.3)$$

the **inner dilatation** of f at x by

$$K_I(x) = K_I(x, f) = \begin{cases} \frac{|J(x, f)|}{l(f'(x))^n} & \text{if } J(x, f) \neq 0, \\ 1 & \text{if } f'(x) = 0, \\ \infty & \text{otherwise,} \end{cases} \quad (4.4)$$

Note that, see, e.g., Section 1.2.1 in [41],

$$K_O(x, f) \leq K_I^{n-1}(x, f) \quad \text{and} \quad K_I(x, f) \leq K_O^{n-1}(x, f), \quad (4.5)$$

in particular, $K_O(x, f) < \infty$ a.e. if and only if $K_I(x, f) < \infty$ a.e. The latter is equivalent to the condition that a.e. either $\det f'(x) > 0$ or $f'(x) = 0$.

Now, recall that a homeomorphism f between domains D and D' in \mathbb{R}^n , $n \geq 2$, is called of **finite distortion** if $f \in W_{\text{loc}}^{1,1}$ and

$$\|f'(x)\|^n \leq K(x) \cdot J_f(x) \quad (4.6)$$

with some a.e. finite function K . The term is due to Tadeusz Iwaniec. In other words, (4.6) just means that dilatations $K_O(x, f)$ and $K_I(x, f)$ are finite a.e.

In view of (4.5), the next statement says on a stronger modulus estimate than the obtained in [23], Theorem 4.1, in terms of the outer dilatation $K_O(x, f)$. It is key for deriving consequences from our theory of lower Q –homeomorphisms.

Theorem 4.1. *Let D and D' be domains in \mathbb{R}^n , $n \geq 3$, and let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nondecreasing function such that, for some $t_* \in \mathbb{R}^+$,*

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \quad (4.7)$$

Then each homeomorphism $f : D \rightarrow D'$ of finite distortion in the class $W_{\text{loc}}^{1,\varphi}$ is a lower Q -homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = [K_I(x, f)]^{\frac{1}{n-1}}$.

Proof. Let B be a (Borel) set of all points $x \in D$ where f has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun's theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [9], we see that B is the union of a countable collection of Borel sets B_l , $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., 3.2.2 as well as 3.1.4 and 3.1.8 in [9]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the rest of all points $x \in D$ where f has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \cup B_*)$ has Lebesgue measure zero, see Theorem 1 in [24]. Hence $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in \mathbb{R}^n and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$, see Theorem 2.11 in [21] or Theorem 9.1 in [30]. Thus, by Corollary 4 in [24] $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ as well as $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r , $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0|$, with the domain D . Given $\varrho_* \in \text{adm } f(\Gamma)$ such that $\varrho_* \equiv 0$ outside of $f(D)$, set $\varrho \equiv 0$ outside of D and on $D \setminus B$ and, moreover,

$$\varrho(x) := \Lambda(x) \cdot \varrho_*(f(x)) \quad \text{for } x \in B$$

where

$$\begin{aligned} \Lambda(x) &:= \left[J_f(x) \cdot K_I^{\frac{1}{n-1}}(x, f) \right]^{\frac{1}{n}} = \left[\frac{\det f'(x)}{l(f'(x))} \right]^{\frac{1}{n-1}} = \\ &= [\lambda_2 \cdot \dots \cdot \lambda_n]^{\frac{1}{n-1}} \geq [J_{n-1}(x)]^{\frac{1}{n-1}} \quad \text{for a.e. } x \in B; \end{aligned}$$

here as usual $\lambda_n \geq \dots \geq \lambda_1$ are principal dilatation coefficients of $f'(x)$, see, e.g., Section I.4.1 in [41], and $J_{n-1}(x)$ is the $(n-1)$ -dimensional Jacobian of $f|_{S_r}$ at x where $r = |x - x_0|$, see Section 3.2.1 in [9].

Arguing piecewise on B_l , $l = 1, 2, \dots$, and taking into account Kirszbraun's theorem, by Theorem 3.2.5 on the change of variables in [9], we have that

$$\int_{S_r} \varrho^{n-1} d\mathcal{A} \geq \int_{S_*^r} \varrho_*^{n-1} d\mathcal{A} \geq 1$$

for a.e. S_r and, thus, $\varrho \in \text{ext adm } \Gamma$.

The change of variables on each B_l , $l = 1, 2, \dots$, see again Theorem 3.2.5 in [9], and countable additivity of integrals give also the estimate

$$\int_D \frac{\varrho^n(x)}{K_I^{\frac{1}{n-1}}(x)} dm(x) \leq \int_{f(D)} \varrho_*^n(x) dm(x)$$

and the proof is complete. \square

Corollary 4.1. *Each homeomorphism f with finite distortion in \mathbb{R}^n , $n \geq 3$, of the class $W_{\text{loc}}^{1,p}$ for $p > n - 1$ is a lower Q -homeomorphism at every point $x_0 \in \overline{D}$ with $Q = K_I^{\frac{1}{n-1}}$.*

5 Boundary behavior of Orlicz–Sobolev classes

In this section we assume that $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing function such that, for some $t_* \in \mathbb{R}^+$,

$$\int_{t_*}^{\infty} \left[\frac{t}{\varphi(t)} \right]^{\frac{1}{n-2}} dt < \infty. \quad (5.1)$$

The continuous extension to the boundary of the inverse mappings has a simpler criterion than for the direct mappings. Hence we start from the former. Namely, in view of Theorem 4.1, we have by Theorem 9.1 in [20] or Theorem 9.6 in [30] the next statement.

Theorem 5.1. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be weakly flat. Suppose that f is a homeomorphism of D onto D' in a class $W_{\text{loc}}^{1,\varphi}$ with condition (5.1) and $K_I \in L^1(D)$. Then f^{-1} can be extended to a continuous mapping of $\overline{D'}$ onto \overline{D} .*

However, as it follows from the example in Proposition 6.3 in [30], see also (4.5), any degree of integrability $K_I \in L^q(D)$, $q \in [1, \infty)$, cannot guarantee the extension by continuity to the boundary of the direct mappings.

Also by Theorem 4.1, we have the following consequence of Proposition 3.1.

Theorem 5.2. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism of finite distortion in $W_{\text{loc}}^{1,\varphi}$ with condition (5.1) such that*

$$\int_0^{\delta(x_0)} \frac{dr}{||K_I||^{\frac{1}{n-1}}(x_0, r)} = \infty \quad \forall x_0 \in \partial D \quad (5.2)$$

for some $\delta(x_0) \in (0, d(x_0))$ where $d(x_0) = \sup_{x \in D} |x - x_0|$ and

$$||K_I||(x_0, r) = \int_{D \cap S(x_0, r)} K_I(x, f) d\mathcal{A}.$$

Then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

In particular, as a consequence of Theorem 5.2, we obtain the following generalization of the well-known theorems of Gehring–Martio and Martio–Vuorinen on a homeomorphic extension to the boundary of quasiconformal mappings between the so-called QED domains, see [11] and [32].

Corollary 5.1. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism of finite distortion in the class $W_{\text{loc}}^{1,p}$, $p > n - 1$. If (5.2) holds, then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.*

Lemma 5.1. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism of finite distortion in $W_{\text{loc}}^{1,\varphi}$ with condition (5.1) such that*

$$\int_{D(x_0, \varepsilon, \varepsilon_0)} K_I(x, f) \cdot \psi_{x_0, \varepsilon}^n(|x - x_0|) dm(x) = o(I_{x_0}^n(\varepsilon)) \text{ as } \varepsilon \rightarrow 0 \quad \forall x_0 \in \partial D \quad (5.3)$$

where $D(x_0, \varepsilon, \varepsilon_0) = \{x \in D : \varepsilon < |x - x_0| < \varepsilon_0\}$ for some $\varepsilon_0 \in (0, \delta_0)$, $\delta_0 = d(x_0) = \sup_{x \in D} |x - x_0|$, and $\psi_{x_0, \varepsilon}(t)$ is a family of non-negative measurable

(by Lebesgue) functions on $(0, \infty)$ such that

$$0 < I_{x_0}(\varepsilon) = \int_{\varepsilon}^{\varepsilon_0} \psi_{x_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (5.4)$$

Then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

Proof. Lemma 5.1 is an immediate consequence of Theorems 4.1 and 5.2 taking into account Lemma 3.7 in the work [46], see Lemma 7.4 in the monograph [30], and extending $K_I(x, f)$ by zero outside of D . \square

Choosing in Lemma 5.1 $\psi(t) = 1/(t \log 1/t)$ and applying Lemma 2.1, we obtain the following result.

Theorem 5.3. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism in $W_{\text{loc}}^{1, \varphi}$ with condition (5.1) such that*

$$K_I(x, f) \leq Q(x) \quad \text{a.e. in } D \quad (5.5)$$

for a function $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $Q \in \text{FMO}(x_0)$ for all $x_0 \in \partial D$. Then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

In the following consequences, we assume that $K_I(x, f)$ is extended by zero outside of D .

Corollary 5.2. *In particular, the conclusions of Theorem 5.3 hold if*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{B(x_0, \varepsilon)} K_I(x, f) dm(x) < \infty \quad \forall x_0 \in \partial D. \quad (5.6)$$

Similarly, choosing in Lemma 5.1 the function $\psi(t) = 1/t$, we come to the following more general statement.

Theorem 5.4. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism in $W_{\text{loc}}^{1, \varphi}$ with condition (5.1) such that*

$$\int_{\varepsilon < |x - x_0| < \varepsilon_0} K_I(x, f) \frac{dm(x)}{|x - x_0|^n} = o\left(\left[\log \frac{\varepsilon_0}{\varepsilon}\right]^n\right) \quad \forall x_0 \in \partial D \quad (5.7)$$

as $\varepsilon \rightarrow 0$ for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \delta(x_0) = \sup_{x \in D} |x - x_0|$. Then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

Corollary 5.3. *The condition (5.7) and the assertion of Theorem 5.4 hold if*

$$K_I(x, f) = o \left(\left[\log \frac{1}{|x - x_0|} \right]^{n-1} \right) \quad (5.8)$$

as $x \rightarrow x_0$. The same holds if

$$k_f(r) = o \left(\left[\log \frac{1}{r} \right]^{n-1} \right) \quad (5.9)$$

as $r \rightarrow 0$ where $k_f(r)$ is the mean value of the function $K_I(x, f)$ over the sphere $|x - x_0| = r$.

Remark 5.1. Choosing in Lemma 5.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (5.7) by

$$\int_{\varepsilon < |x - x_0| < 1} \frac{K_I(x, f) dm(x)}{\left(|x - x_0| \log \frac{1}{|x - x_0|} \right)^n} = o \left(\left[\log \log \frac{1}{\varepsilon} \right]^n \right) \quad (5.10)$$

and (5.9) by

$$k_f(r) = o \left(\left[\log \frac{1}{r} \log \log \frac{1}{r} \right]^{n-1} \right). \quad (5.11)$$

Thus, it is sufficient to require that

$$k_f(r) = O \left(\left[\log \frac{1}{r} \right]^{n-1} \right) \quad (5.12)$$

In general, we could give here the whole scale of the corresponding conditions in terms of log using functions $\psi(t)$ of the form $1/(t \log \dots \log 1/t)$.

Theorem 5.5. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 3$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a homeomorphism in $W_{\text{loc}}^{1,\varphi}$ with condition (5.1) such that*

$$\int_D \Phi(K_I(x, f)) dm(x) < \infty \quad (5.13)$$

for a non-decreasing convex function $\Phi : \overline{\mathbb{R}^+} \rightarrow \overline{\mathbb{R}^+}$. If, for some $\delta > \Phi(0)$,

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \quad (5.14)$$

then f can be extended to a (continuous) homeomorphic map $\bar{f} : \overline{D} \rightarrow \overline{D'}$.

Indeed, by Theorem 3.1 and Corollary 3.2 in [48], (5.13) and (5.14) imply (5.2) and, thus, Theorem 5.5 is a direct consequence of Theorem 5.2.

Corollary 5.4. *The conclusion of Theorem 5.5 holds if, for some $\alpha > 0$,*

$$\int_D e^{\alpha K_I(x,f)} dm(x) < \infty. \quad (5.15)$$

Remark 5.2. Note that by Theorem 5.1 and Remark 5.1 in [22] the conditions (5.14) are not only sufficient but also necessary for continuous extension to the boundary of f with the integral constraint (5.13).

Recall that by Theorem 2.1 in [48], see also Proposition 2.3 in [47], condition (5.14) is equivalent to a series of other conditions and, in particular, to the following condition

$$\int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^{n'}} = +\infty \quad (5.16)$$

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e., $n' = 2$ for $n = 2$, n' is strictly decreasing in n and $n' = n/(n-1) \rightarrow 1$ as $n \rightarrow \infty$.

Finally, note that all these results hold, for instance, if $f \in W_{\text{loc}}^{1,p}$, $p > n-1$.

6 On finitely bi-Lipschitz mappings

Given an open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, following Section 5 in [21], see also Section 10.6 in [30], we say that a mapping $f : \Omega \rightarrow \mathbb{R}^n$ is **finitely bi-Lipschitz** if

$$0 < l(x, f) \leq L(x, f) < \infty \quad \forall x \in \Omega \quad (6.1)$$

where

$$L(x, f) = \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \quad (6.2)$$

and

$$l(x, f) = \liminf_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|}. \quad (6.3)$$

By the classic Rademacher–Stepanov theorem, we obtain from the right hand inequality in (6.1) that finitely bi-Lipschitz mappings are differentiable a.e. and from the left hand inequality in (6.1) that $J_f(x) \neq 0$ a.e. Moreover, such mappings have (N) –property with respect to each Hausdorff measure, see, e.g., either Lemma 5.3 in [21] or Lemma 10.6 [30]. Thus, the proof of the following theorems is perfectly similar to one of Theorem 4.1 and hence we omit it, cf. also similar but weaker Corollary 5.15 in [21] and Corollary 10.10 in [30] formulated in terms of the outer dilatation K_O .

Theorem 6.1. *Every finitely bi-Lipschitz homeomorphism $f : \Omega \rightarrow \mathbb{R}^n$, $n \geq 2$, is a lower Q -homeomorphism with $Q = K_I^{\frac{1}{n-1}}$.*

All results for finitely bi-Lipschitz homeomorphisms are perfectly similar to the corresponding results for homeomorphisms with finite distortion in the Orlich–Sobolev classes from the last section. Hence we will not formulate all them in the explicit form here in terms of inner dilatation K_I .

We give here for instance only one of these results.

Theorem 6.2. *Let D and D' be bounded domains in \mathbb{R}^n , $n \geq 2$, D be locally connected on ∂D and $\partial D'$ be (strongly accessible) weakly flat. Suppose that $f : D \rightarrow D'$ is a finitely bi-Lipschitz homeomorphism such that*

$$\int_D \Phi(K_I(x, f)) \, dm(x) < \infty \quad (6.4)$$

for a non-decreasing convex function $\Phi : \overline{\mathbb{R}^+} \rightarrow \overline{\mathbb{R}^+}$. If, for some $\delta > \Phi(0)$,

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{n-1}}} = \infty \quad (6.5)$$

then f can be extended to a (continuous) homeomorphic map $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

Corollary 6.1. *The conclusion of Theorem 6.2 holds if, for some $\alpha > 0$,*

$$\int_D e^{\alpha K_I(x, f)} \, dm(x) < \infty. \quad (6.6)$$

References

- [1] Andreian Cazacu C., *Influence of the orientation of the characteristic ellipses on the properties of the quasiconformal mappings*, Proceedings of Romanian-Finnish Seminar on Teichmüller Spaces and Riemann Surfaces, Bucharest, 1971, 65-85.
- [2] Andreian Cazacu C., *Moduli inequalities for quasiregular mappings*. Ann. Acad. Sci. Fenn. Math. **2** (1976), 17–28.
- [3] Andreian Cazacu C., *On the length-area dilatation*. Complex Var. Theory Appl. **50** (2005), no. 7–11, 765–776.
- [4] BIRNBAUM Z., ORLICZ W. Über die Verallgemeinerungen des Begriffes der zueinander konjugierten Potenzen // Studia. Math. – 1931. – **3**. – P. 1–67.
- [5] CHIARENZA F., FRASCA M., LONGO P. $W^{2,p}$ -solvability of the Dirichlet problem for non-divergence elliptic equations with VMO coefficients // Trans. Amer. Math. Soc. – 1993. – **336**, no. 2. – P. 841–853.
- [6] Cristea M., *Local homeomorphisms satisfying generalized modular inequalities*. Complex Var. Elliptic Equ. **59** (2014), no. 10, 1363–1387.
- [7] Cristea M., *On generalized quasiconformal mappings*. Complex Var. Elliptic Equ. **59** (2014), no. 2, 232–246.
- [8] Cristea M., *On generalized quasiregular mappings*. Complex Var. Elliptic Equ. **58** (2013), no. 12, 1745–1764.
- [9] FEDERER H. Geometric Measure Theory. – Berlin: Springer-Verlag, 1969.
- [10] GEHRING F.W. Rings and quasiconformal mappings in space // Trans. Amer. Math. Soc. – 1962. – **103**, no. 3. – P. 353 – 393.
- [11] GEHRING F.W., MARTIO O. Quasiextremal distance domains and extension of quasiconformal mappings // J. Anal. Math. – 1985. – **45**. – P. 181–206.
- [12] GUTLYANSKII V., RYAZANOV V., SREBRO U., YAKUBOV E. The Beltrami Equations: A Geometric Approach. – Developments in Math. **26**. – New York etc.: Springer, 2012.
- [13] HEINONEN J., KILPELAINEN T., MARTIO O. Nonlinear Potential Theory of Degenerate Elliptic Equations. – Oxford-New York-Tokio: Clarendon Press, 1993.
- [14] IGNAT'EV A., RYAZANOV V. Finite mean oscillation in the mapping theory // Ukr. Mat. Vis. – 2005. – **2**, no. 3. – P. 395–417, 443 [in Russian]; transl. in Ukr. Math. Bull. – 2005. – **2**, no. 3. – P. 403–424, 443.
- [15] IWANIEC T., SBORDONE C. Riesz transforms and elliptic PDEs with VMO coefficients // J. Anal. Math. – 1998. – **74**. – P. 183–212.

- [16] John F. and Nirenberg L.: On functions of bounded mean oscillation // *Comm. Pure Appl. Math.* – 1961. – **14**. – P. 415–426.
- [17] KOVTONYUK D., PETKOV I., RYAZANOV V. On the boundary behaviour of solutions to the Beltrami equations // *Complex Variables and Elliptic Equations.* – 2013. – **58**, no. 5. – P. 647 – 663.
- [18] KOVTONYUK D.A., PETKOV I.V., RYAZANOV V.I., SALIMOV R.R. Boundary behavior and Dirichlet problem for Beltrami equations // *Algebra and Analysis.* - 2013. - **25**, no. 4. - P. 101-124 [in Russian]; transl in *St. Petersburg Math. J.* - 2014. - **25**. - P. 587- 603.
- [19] KOVTONYUK D.A., RYAZANOV V.I. On boundaries of space domains // *Proc. Inst. Appl. Math. & Mech. NAS of Ukraine.* – 2006. – **13**. – P. 110 – 120.
- [20] KOVTONYUK D.A., RYAZANOV V.I. Toward the theory of lower Q -homeomorphisms // *Ukr. Mat. Visn.* – 2008. – **5**, no. 2. – P. 159 – 184 [in Russian]; transl. in *Ukrainian Math. Bull.* – 2008. – **5**, no. 2. – P. 157 – 181.
- [21] KOVTONYUK D., RYAZANOV V. On the theory of mappings with finite area distortion // *J. Anal. Math.* – 2008. – **104**. – P. 291–306.
- [22] KOVTONYUK D., RYAZANOV V. On the boundary behavior of generalized quasi-isometries // *J. Anal. Math.* – 2011. – **115**. – P. 103–119.
- [23] KOVTONYUK D.A., RYAZANOV V.I., SALIMOV R.R., SEVOST'YANOV E.A. On mappings in the Orlicz-Sobolev classes // *Ann. Univ. Bucharest, Ser. Math* - 2012. - **3(LXI)**, no. 1. - P. 67-78.
- [24] KOVTONYUK D.A., RYAZANOV V.I., SALIMOV R.R., SEVOST'YANOV E.A. Toward the theory of classes of Orlicz–Sobolev // *Algebra and Analysis.* – 2013. – **25**, no. 6. – P. 49 – 101 [in Russian]; transl in *St. Petersburg Math. J.*
- [25] KOVTONYUK D.A., SALIMOV R.R., SEVOST'YANOV E.A. (ED. RYAZANOV V.I.) Toward the Theory of Mappings in Classes of Sobolev and Orlicz–Sobolev. – Kiev: Naukova dumka, 2013 [in Russian].
- [26] MARTIO O., RICKMAN S., VAISALA J. Definitions for quasiregular mappings // *Ann. Acad. Sci. Fenn. Ser. A1. Math.* – 1969. – **448**. – P. 1–40.
- [27] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Mappings with finite length distortion // *J. Anal. Math.* – 2004. – **93**. – P. 215–236.
- [28] Martio O., Ryazanov V., Srebro U. and Yakubov E.: Q -homeomorphisms // *Contemporary Math.* – 2004. – **364**. – P. 193–203.
- [29] Martio O., Ryazanov V., Srebro U., Yakubov E.: On Q -homeomorphisms // *Ann. Acad. Sci. Fenn. Ser. A1. Math.* – 2005. – **30**. – P. 49–69.

- [30] MARTIO O., RYAZANOV V., SREBRO U., YAKUBOV E. *Moduli in Modern Mapping Theory*. – Springer Monographs in Mathematics. – New York etc.: Springer, 2009.
- [31] MARTIO O., RYAZANOV V., VUORINEN M. BMO and Injectivity of Space Quasiregular Mappings // *Math. Nachr.* – 1999. – **205**. – P. 149–161.
- [32] MARTIO O., VUORINEN M. Whitney cubes, p -capacity and Minkowski content // *Expo. Math.* – 1987. – **5**. – P. 17–40.
- [33] MAZ'YA V. *Sobolev Spaces*. – Leningrad: Leningrad. Univ., 1985; transl. in Berlin: Springer-Verlag, 1985.
- [34] Näkki R.: Boundary behavior of quasiconformal mappings in n -space // *Ann. Acad. Sci. Fenn. Ser. A1. Math.* – 1970. – **484**. – P. 1–50.
- [35] OHTSUKA M. *Extremal Length and Precise Functions*. – Tokyo: Gakkotosho Co., 2003.
- [36] ORLICZ W. Über eine gewisse Klasse von Räumen vom Typus B // *Bull. Intern. de l'Acad. Pol. Serie A, Cracovie*. – 1932.
- [37] PALAGACHEV D.K. Quasilinear elliptic equations with VMO coefficients // *Trans. Amer. Math. Soc.* – 1995. – **347**, no. 7. – P. 2481–2493.
- [38] RADO T., REICHELDERFER P.V. *Continuous Transformations in Analysis*. – Berlin etc.: Springer, 1955.
- [39] RAGUSA M.A. Elliptic boundary value problem in vanishing mean oscillation hypothesis // *Comment. Math. Univ. Carolin.* – 1999. – **40**, no. 4. – P. 651–663.
- [40] REIMANN H.M., RYCHENER T. *Funktionen Beschränkter Mittlerer Oscillation*. – Lecture Notes in Math. **487**. – Berlin etc.: Springer-Verlag, 1975.
- [41] RESHETNYAK YU.G. *Space mappings with bounded distortion*. – Novosibirsk: Nauka, 1982; transl. in *Translations of Mathematical Monographs*, **73**. – Providence, RI: Amer. Math. Soc., 1988.
- [42] Reshetnyak Yu.G.: Sobolev classes of functions with values in a metric space. *Sibirsk. Mat. Zh.* – 1997. – **38**. – 657–675.
- [43] RICKMAN S. *Quasiregular Mappings*. – Berlin etc.: Springer-Verlag, 1993.
- [44] RYAZANOV V.I., SALIMOV R.R. Weakly flat spaces and boundaries in the mapping theory // *Ukr. Mat. Vis.* – 2007. – **4**, № 2. – P. 199 – 234 [in Russian]; transl. in *Ukrainian Math. Bull.* – 2007. – **4**, no. 2. – P. 199 – 233.
- [45] RYAZANOV V., SALIMOV R., SREBRO U., YAKUBOV E. On Boundary Value Problems for the Beltrami Equations // *Contemporary Math.* – 2013. – **591**. – P. 211–242.

- [46] V. Ryazanov and E. Sevost'yanov, *Equicontinuous classes of ring Q -homeomorphisms* // Sibirsk. Mat. Zh. – 2007. – **48**, no. 6. – P. 1361–1376; translation in Siberian Math. J. – 2007. – **48**, no. 6. – P. 1093–1105.
- [47] RYAZANOV V., SEVOST'YANOV E. Equicontinuity of mappings quasiconformal in the mean // Ann. Acad. Sci. Fenn. – 2011. – **36**. – P. 231 – 244.
- [48] RYAZANOV V., SREBRO U., YAKUBOV E. On integral conditions in the mapping theory // Ukrainian Math. Bull. – 2010. – **7**, no. 1. – P. 73–87.
- [49] SAKS S. Theory of the Integral. – New York: Dover Publications Inc., 1964.
- [50] SARASON D. Functions of vanishing mean oscillation // Trans. Amer. Math. Soc. – 1975. – **207**. – P. 391–405.
- [51] SOBOLEV S.L. Applications of functional analysis in mathematical physics. – Leningrad: Izdat. Gos. Univ., 1950; transl. in Providence, R.I.: Amer. Math. Soc., 1963.
- [52] VASIL'EV A. Moduli of Families of Curves for Conformal and Quasiconformal Mappings. – Lecture Notes in Math. **1788**. – Berlin etc.: Springer–Verlag, 2002.
- [53] VÄISÄLÄ J. Lectures on n -Dimensional Quasiconformal Mappings. – Lecture Notes in Math. **229**. – Berlin etc.: Springer–Verlag, 1971.
- [54] VODOP'YANOV S. Mappings with bounded distortion and with finite distortion on Carnot groups // Sibirsk. Mat. Zh. – 1999. – **40**, no. 4. – P. 764–804; transl. in Siberian Math. J. – 1999. – **40**, no. 4. – P. 644–677.
- [55] VUORINEN M. Conformal Geometry and Quasiregular Mappings. – Lecture Notes in Math. **1319**. – Berlin etc.: Springer–Verlag, 1988.
- [56] WHYBURN G.TH. Analytic Topology. – Providence: AMS, 1942.
- [57] WILDER R.L. Topology of Manifolds. – New York: AMS, 1949.
- [58] ZAAANEN A.C. Linear analysis. – Noordhoff (1953).

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